

# Inverse problem on a tree-shaped network

Lucie Baudouin\*, Masahiro Yamamoto†

July 22, 2014

## Abstract

In this article, we prove a uniqueness result for a coefficient inverse problems regarding a wave, a heat or a Schrödinger equation set on a tree-shaped network, as well as the corresponding stability result of the inverse problem for the wave equation. The objective is the determination of the potential on each edge of the network from the additional measurement of the solution at all but one external end-points. Our idea for proving the uniqueness is to use a traditional approach in coefficient inverse problem by Carleman estimate. Afterwards, using an observability estimate on the whole network, we apply a compactness-uniqueness argument and prove the stability for the wave inverse problem.

**Keywords:** networks, inverse problem, Carleman estimate.

**AMS subject classifications:** 35R30, 93C20, 34B45

## 1 Introduction and main results

Let  $\Lambda$  be a tree-shaped network composed of  $N + 1$  open segments  $(e_j)_{j=0,1,\dots,N}$  of length  $\ell_j$ , linked by  $N_1$  internal node points belonging to the set  $\Pi_1$  and let us denote by  $\Pi_2$  the set of  $N_2$  exterior end-points where only one segment starts. Here we note that  $N + 1 = N_1 + N_2$ . By “tree-shaped network”, we mean that  $\Lambda$  does not contain any closed loop.

For any function  $f : \Lambda \rightarrow \mathbb{R}$  and any internal node  $P \in \Pi_1$  where  $n_P$  segments, say  $e_1, \dots, e_{n_P}$ , meet, we set

$$f_j = f|_{e_j} : \text{the restriction of } f \text{ to the edge } e_j, \text{ and } [f]_P := \sum_{j=1}^{n_P} f_j(P).$$

We consider on this plane 1-d tree-shaped network  $\Lambda$  either wave or heat or even Schrödinger equations, with a different potential term  $x \mapsto p_j(x)$  on each segment.

Our first, and main, system of interest is the following 1-d wave equation on the network  $\Lambda$ :

$$\begin{cases} \partial_t^2 u_j - \partial_x^2 u_j + p_j(x)u_j = 0 & \forall j \in \{0, 1, \dots, N\}, (x, t) \in e_j \times (0, T), \\ u(Q, t) = h(t), & \forall Q \in \Pi_2, t \in (0, T), \\ u(x, 0) = u^0(x), \partial_t u(x, 0) = u^1(x), & x \in \Lambda, \end{cases} \quad (1)$$

assuming some compatibility condition between the boundary and initial data. Moreover we assume the continuity and what is called the Kirchhoff law at any internal node  $P \in \Pi_1$ , which are given by

$$u_j(P, t) = u_i(P, t) =: u(P, t), \quad \forall i, j \in \{1, \dots, n_P\}, 0 < t < T, \quad (2)$$

$$[u_x(t)]_P := \sum_{j=1}^{n_P} \partial_x u_j(P, t) = 0, \quad 0 < t < T. \quad (3)$$

---

\*CNRS ; LAAS ; 7 avenue du colonel Roche, F-31077 Toulouse, France ;  
 Université de Toulouse ; UPS, INSA, INP, ISAE, UT1, UTM, LAAS ; F-31077 Toulouse, France.  
 E-mail: [lucie.baudouin@laas.fr](mailto:lucie.baudouin@laas.fr)

†Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914 Japan.  
 E-mail: [myama@next.odn.ne.jp](mailto:myama@next.odn.ne.jp)

Henceforth we choose an orientation of  $\Lambda$  such that to two endpoints of each segment  $e$ , correspond an initial node  $I(e)$  and a terminal node  $T(e)$ . We further define the outward normal derivative  $\partial_{n_e} u_j$  at a node  $P$  of  $e_j$  by

$$\partial_{n_e} u_j(P, t) = \begin{cases} -\partial_x u_j(P, T), & \text{if } P \in I(e_j), \\ \partial_x u_j(P, T), & \text{if } P \in T(e_j). \end{cases}$$

Henceforth we set

$$u = (u_0, \dots, u_N), \quad u_j = u|_{e_j}, \quad \text{and} \quad p = (p_0, \dots, p_N), \quad p_j = p|_{e_j} \quad \text{for } j \in \{0, 1, \dots, N\}.$$

Let us also mention that at a node point, at least three segments  $e_j$  meet. If only two segments, say  $e_1, e_2$ , meet at a node point, then by (2) and (3), setting  $u = u_1$  and  $p = p_1$  in  $e_1$  and  $u = u_2$ ,  $p = p_2$  in  $e_2$ , we have  $\partial_t^2 u - \partial_x^2 u + pu$  in  $e_1 \cup e_2$ . Therefore we can regard  $e_1 \cup e_2$  as one open segment.

Since one can prove the unique existence of solution to (1) - (3) in a suitable function space (e.g., Lions and Magenes [17]), we denote the solution by  $u[p](x, t)$ , and we set  $u[p] = (u[p]_0, \dots, u[p]_N)$ .

Moreover we consider the following heat system on the same network  $\Lambda$

$$\begin{cases} \partial_t u_j - \partial_x^2 u_j + p_j(x) u_j = 0 & \forall j \in \{0, 1, \dots, N\}, \forall (x, t) \in e_j \times (0, T), \\ \partial_x u(Q, t) = 0, & \forall Q \in \Pi_2, \forall t \in (0, T), \\ u(x, 0) = u^0(x), & \forall x \in \Lambda, \end{cases} \quad (4)$$

and the Schrödinger system on the network  $\Lambda$

$$\begin{cases} i\partial_t u_j - \partial_x^2 u_j + p_j(x) u_j = 0 & \forall j \in \{0, 1, \dots, N\}, \forall (x, t) \in e_j \times (0, T), \\ u(Q, t) = h(t), & \forall Q \in \Pi_2, \forall t \in (0, T), \\ u(x, 0) = u^0(x), & \forall x \in \Lambda, \end{cases} \quad (5)$$

both under the same node conditions (2) and (3). Here and henceforth we set  $i = \sqrt{-1}$ . If there is no possible confusion, by the same notation  $u[p]$  we denote the solution to (4) or (5), under (2) and (3).

**Inverse Problem:** Is it possible to retrieve the potential  $p$  everywhere in the whole network  $\Lambda$  from measurements at all external nodes except one?

In our article, we address the following two fundamental theoretical questions concerning coefficient inverse problems:

**Uniqueness:** Do the equalities of the measurements  $\partial_x u[p](Q, t) = \partial_x u[q](Q, t)$  for all  $t \in (0, T)$  and  $Q \in \Pi_2 \setminus \{Q_{N_2}\}$  imply  $p = q$  on  $\Lambda$ ?

**Stability:** Can we estimate, in appropriate norms, the difference of two potentials  $p - q$  on  $\Lambda$  by the difference of the corresponding measurements  $\partial_x u[p](Q, t) - \partial_x u[q](Q, t)$  for all  $t \in (0, T)$  and  $Q \in \Pi_2 \setminus \{Q_{N_2}\}$ ?

This inverse problem is nonlinear and we will give here the proof of the uniqueness of the solution with an argument which do not use a global Carleman estimate. Very recent papers on coefficient inverse problems on networks, as Baudouin, Crépeau and Valein [1] for the wave equation, and Ignat, Pazoto and Rosier [8] for the heat and the Schrödinger equations, give indeed *stability* and therefore *uniqueness* from appropriate global Carleman estimates. Our first goal is to prove the uniqueness of the potential on the tree-shaped network from measurements only at all the exterior end-points of the network, except one. The argument for the uniqueness will work for either the wave or the heat or the Schrödinger equations on the network. The question of the proof of the Lipschitz stability in the case of the wave equation will be addressed afterwards, using a compactness-uniqueness argument, and relies on the observability estimate on the whole network which was already proved in the literature in several situations.

Concerning the precise topic which we are considering, the bibliography lies in two different domains, namely coefficient inverse problems for partial differential equation on the one hand and control and stabilization in networks on the other hand.

Therefore one can begin by mentioning the book of Isakov [11] which addresses some techniques linked to the study of inverse problem for several partial differential equations. Actually, as the

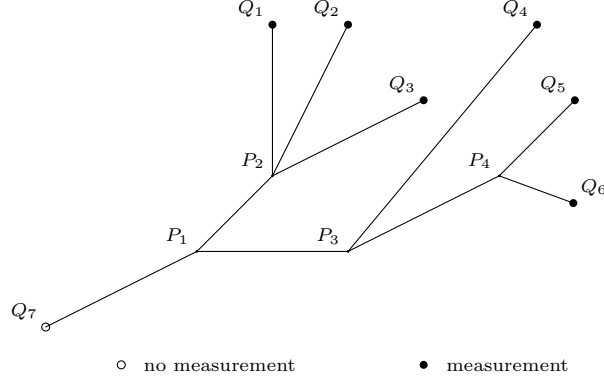


Figure 1: A star-shaped network with 10 edges ( $N = 9$ ,  $N_1 = 4$ ,  $N_2 = 7$ ).

first answer to the uniqueness for a coefficient inverse problem with a single measurement, we refer to Bukhgeim and Klivanov [4], and see also Klivanov [12] and Yamamoto [24] for example. Here we do not intend to give an exhaustive list of references. After the proof of uniqueness using the basic 1-d result on the basis of local Carleman estimates, the idea beneath this article is to take advantage of an observability estimate to obtain the Lipschitz stability of the inverse problem with a compactness-uniqueness argument. Nowadays, many results on the stability of inverse problems are derived directly from global Carleman estimates, and see e.g., [1] and [8]. One should also know that studies on inverse problems and controllability of partial differential equations share some technical materials such as Carleman estimates and observability inequalities. In the particular network setting, we would like to make use of classical results such as well-known 1-d local Carleman estimates, observability estimates on the network borrowed from control studies, in order to obtain uniqueness and stability results. We can also give some more references on inverse problems for hyperbolic equations such as Baudouin, Mercado and Osses [2], Imanuvilov and Yamamoto [9], [10], Puel and Yamamoto [20], Yamamoto and Zhang [25], which are all based upon local or global Carleman estimates.

Besides, the control, observation and stabilization problems of networks have been the object of recent and intensive researches such as e.g., Dager and Zuazua [7], Lagnese, Leugering and Schmidt [14], Zuazua [28]. More specifically, the control being only applied at one single end of the network, the articles Dager [5], Dager and Zuazua [6, 7] prove controllability results for the wave equation on networks, using observability inequalities under assumptions about the irrationality properties of the ratios of the lengths of the strings. We can also underline that many results of controllability on networks concern only the wave equation without lower order terms (see [14], Schmidt [22] for instance). However it is difficult to consider such measurements at more limited nodes for the inverse problem and we do not consider the measurements at less external nodes.

In the sequel, we shall use the following notations:

$$\begin{aligned}
 L^\gamma(\Lambda) &= \{f; f_j \in L^\gamma(e_j), \forall j \in \{0, 1, \dots, N\}\}, \quad \gamma \geq 1, \\
 H_0^1(\Lambda) &= \left\{ f; f_j \in H^1(e_j), \forall j \in \{0, 1, \dots, N\}, f_j(P) = f_k(P) \text{ if } e_j \text{ and } e_k \text{ meet at } P, \right. \\
 &\quad \left. \forall P \in \Pi_1, \text{ and } f(Q) = 0, \forall Q \in \Pi_2 \right\}.
 \end{aligned}$$

For shortness, for  $f \in L^1(\Lambda)$ , we often write,

$$\int_{\Lambda} f dx = \sum_{j=0}^N \int_{e_j} f_j(x) dx,$$

where the integral on  $e_j$  is oriented from  $I(e_j)$  to  $T(e_j)$ . Then the norms of the Hilbert spaces

$L^2(\Lambda)$  and  $H_0^1(\Lambda)$  are defined by

$$\|f\|_{L^2(\Lambda)}^2 = \int_{\Lambda} |f|^2 dx \text{ and } \|f\|_{H_0^1(\Lambda)}^2 = \int_{\Lambda} |\partial_x f|^2 dx.$$

For  $M \geq 0$ , we introduce the set

$$L_M^\infty(\Lambda) = \{q = (q_0, \dots, q_N); q_j \in L^\infty(e_j), \forall j \in \{0, 1, \dots, N\} \text{ such that } \|q\|_{L^\infty(\Lambda)} \leq M\}.$$

We are ready to state our first main result:

**Theorem 1 (Uniqueness)** *Let  $r > 0$  be an arbitrary constant. Assume that  $p, q \in L^\infty(\Lambda)$  and the initial value  $u^0$  satisfies*

$$|u^0(x)| \geq r > 0, \quad \text{a.e. in } \Lambda.$$

*Assume further that the solutions  $u[p], u[q]$  of (1)-(2)-(3) belong to*

$$H^3(0, T; L^\infty(\Lambda)) \cap H^1(0, T; H^2(\Lambda)).$$

*Then there exists  $T_0 > 0$  such that for all  $T \geq T_0$ , if*

$$\partial_x u[p](Q, t) = \partial_x u[q](Q, t) \quad \text{for each } t \in (0, T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have  $p = q$  in  $\Lambda$ .*

The proof of this result in Section 2 relies on a 1-d result of uniqueness for the determination of potential in the wave equation and an “undressing” argument.

It is worth mentioning that our argument gives the uniqueness for the inverse problems of determination of potentials on tree-shaped networks also for the heat and the Schrödinger equations using only measurements at  $N_2 - 1$  exterior end-points. In fact, our arguments in proving the uniqueness for the wave and the Schrödinger equations are essentially the same and are based on local Carleman estimates, while the uniqueness for the inverse heat problem is reduced to the uniqueness for the corresponding inverse wave problem (in a sense to be detailed later).

**Theorem 2 (Uniqueness for the heat inverse problem)** *Assume that  $p, q \in L^\infty(\Lambda)$ , the initial value  $u^0$  satisfies*

$$|u^0(x)| \geq r > 0, \quad \text{a.e. in } \Lambda$$

*for some constant  $r$ , and the solutions  $u[p]$  and  $u[q]$  to (4)-(2)-(3), belong to*

$$H^2(0, T; L^\infty(\Lambda)) \cap H^1(0, T; H^2(\Lambda)).$$

*Then there exists  $T > 0$  such that if*

$$u[p](Q, t) = u[q](Q, t) \quad \text{for each } t \in (0, T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have  $p = q$  in  $\Lambda$ .*

**Theorem 3 (Uniqueness for the Schrödinger inverse problem)** *Assume that  $p, q \in L^\infty(\Lambda)$ , the initial value  $u^0$  satisfies*

$$|u^0(x)| \geq r > 0, \quad \text{a.e. in } \Lambda$$

*for some constant  $r$ , and the solutions  $u[p]$  and  $u[q]$  to (5)-(2)-(3), belong to*

$$H^2(0, T; L^\infty(\Lambda)) \cap H^1(0, T; H^2(\Lambda)).$$

*Then there exists  $T > 0$  such that*

$$\partial_x u[p](Q, t) = \partial_x u[q](Q, t) \quad \text{for each } t \in (0, T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have  $p = q$  in  $\Lambda$ .*

One can refer to [1] for the same inverse problem in the wave equation on a network where the proof is detailed in a star-shaped network but is actually generalizable to tree-shaped networks. Reference [8] discusses the inverse heat problem on tree-shaped network. Moreover the paper [8] treats the Schrödinger case in a star-shaped network and needs measurements at all external nodes. We do not know any uniqueness result for non-tree graphs, which are graphs containing a closed cycle. For observability inequality on general graph, see e.g., [7].

For the inverse problem in the wave equation case, we state

**Theorem 4 (Stability)** Let  $M > 0$  and  $r > 0$ . Assume that  $p \in L_M^\infty(\Lambda)$  and the solutions  $u[p]$  and  $u[q]$  to (1)-(2)-(3) satisfy

$$u[p], u[q] \in H^3(0, T; L^\infty(\Lambda)) \cap H^1(0, T; H^2(\Lambda)).$$

Assume also that the initial value  $u^0$  satisfies

$$|u^0(x)| \geq r > 0, \quad \text{a.e. in } \Lambda.$$

Then there exists  $T_0 > 0$  such that for all  $T \geq T_0$ , there exists  $C = C(T, r, M, \ell_0, \dots, \ell_N) > 0$  such that

$$\|q - p\|_{L^2(\Lambda)} \leq C \sum_{j=1}^{N_2-1} \|\partial_x u_j[p](Q_j) - \partial_x u_j[q](Q_j)\|_{H^1(0, T)}. \quad (6)$$

This paper is composed of five sections. The proof of uniqueness in the inverse problem in the wave equation case (Theorem 1) is presented in Section 2. The cases of Schrödinger and heat equations are studied in Section 3, devoted to the proofs of Theorems 2 and 3. Theorem 4 is finally proven in Section 5 by a compactness-uniqueness argument and an observability estimate on the whole network.

We conclude this section with a classical result on the existence and regularity of solutions of the wave system and provide the corresponding energy estimates for the solution which we will need later.

**Lemma 1** Let  $\Lambda$  be a tree-shaped network and assume that  $p \in L_M^\infty(\Lambda)$ ,  $g \in L^1(0, T; L^2(\Lambda))$ ,  $u^0 \in H_0^1(\Lambda)$  and  $u^1 \in L^2(\Lambda)$ . We consider the 1-d wave equation on the network with the conditions (2) and (3):

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + p(x)u = g(x, t), & \text{in } \Lambda \times (0, T), \\ u(Q, t) = 0, & \text{in } (0, T), Q \in \Pi_2, \\ u_j(P, t) = u_k(P, t), & \text{in } (0, T), P \in \Pi_1, j, k \in \{1, \dots, n_P\}, \\ [\partial_x u(t)]_P = 0, & \text{in } (0, T), P \in \Pi_1, \\ u(0) = u^0, \quad \partial_t u(0) = u^1, & \text{in } \Lambda. \end{cases} \quad (7)$$

The Cauchy problem is well-posed and equation (7) admits a unique weak solution

$$u \in C([0, T], H_0^1(\Lambda)) \cap C^1([0, T], L^2(\Lambda)).$$

Moreover there exists a constant  $C = C(\Lambda, T, M) > 0$  such that for all  $t \in (0, T)$ , the energy

$$E(t) = \|\partial_t u(t)\|_{L^2(\Lambda)}^2 + \|\partial_x u(t)\|_{L^2(\Lambda)}^2$$

of the system (7) satisfies

$$E(t) \leq C \left( \|u^0\|_{H_0^1(\Lambda)}^2 + \|u^1\|_{L^2(\Lambda)}^2 + \|g\|_{L^1(0, T, L^2(\Lambda))}^2 \right) \quad (8)$$

and we also have the following trace estimate

$$\sum_{j=1}^{N_2} \|\partial_x u_j(Q_j)\|_{L^2(0, T)}^2 \leq C \left( \|u^0\|_{H_0^1(\Lambda)}^2 + \|u^1\|_{L^2(\Lambda)}^2 + \|g\|_{L^1(0, T, L^2(\Lambda))}^2 \right). \quad (9)$$

The proof of the unique existence of solution to equation (7) can be read in [17, Chap. 3]. Estimate (8) is a classical result which can be formally obtained by multiplying the main equation in (7) by  $\partial_t u_j$ , summing up for  $j \in \{0, \dots, N\}$  the integral of this equality on  $(0, T) \times e_j$  and using some integrations by parts. Estimate (9) is a hidden regularity result which can be obtained by multipliers technique (we refer to [16, Chapter 1]). Formally, for the particular case of a star-shaped network of vertex  $P = 0$  for example, it comes from the multiplication of (7) by  $m(x)\partial_x u_j$ , where  $m \in C^1(\bar{\Lambda})$  with  $m(0) = 0$  and  $m_j(l_j) = 1$ , summing up the integrals of this equality on  $(0, T) \times (0, l_j)$  over  $j \in \{0, \dots, N\}$  and using integrations by parts.

## 2 Uniqueness of the inverse problem - wave network case

As already evoked in the introduction, the proof of Theorem 1 will use a well-known 1-d result of uniqueness for the inverse problem. We recall it in the following lemma.

**Lemma 2** *Let  $r > 0$ ,  $p \in L^\infty(0, \ell)$  and  $T > 2\ell$ . Consider the 1-d wave equation in  $[0, \ell]$  with homogeneous Dirichlet boundary data as follows:*

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p(x)y = f(x)R(x, t), & (x, t) \in (0, \ell) \times (0, T), \\ y(\ell, t) = 0, & t \in (0, T), \\ y(x, 0) = 0, \partial_t y(x, 0) = 0, & x \in (0, \ell), \end{cases} \quad (10)$$

where  $f \in L^2(0, \ell)$  and  $R \in H^1(0, T; L^\infty(0, \ell))$  satisfies  $|R(x, 0)| \geq r > 0$  a.e. in  $(0, \ell)$ . If  $\partial_x y(\ell, t) = 0$  for all  $t \in (0, T)$ , then we have  $f \equiv 0$  in  $(0, \ell)$  and  $y \equiv 0$  in  $(0, \ell) \times (0, T)$ .

This lemma is a classical uniqueness result for the inverse source problem in a wave equation and the proof can be done by the method in [4] on the basis of a 1-d Carleman estimate and the even extension of  $y$  to negative times  $t$ . We further refer to Imanuvilov and Yamamoto [9], [10], Klivanov [12], Klivanov and Timonov [13] for example, and we omit details of the proof.

**Proof of Theorem 1.** We define the following operation of “removing” segments from the tree-shaped network  $\Lambda$ , starting from all the external nodes where we make measurements, except one. We divide the proof into several steps.

**Step 1.** From Lemma 2, we can easily prove that if  $e_j$  is a segment of  $\Lambda$  which ends at an external node  $Q_j \in \Pi_2$ , and if the solutions  $u[p]$  and  $u[q]$  to (1) satisfy  $\partial_x u[p](Q_j, t) = \partial_x u[q](Q_j, t)$  for all  $t \in (0, T)$ , then  $p = q$  on the segment  $e_j$  and  $u[p](x, t) = u[q](x, t)$  for all  $x \in e_j$  and for all  $t \in (0, T)$ . Indeed, if we set  $y = u_j[p_j] - u_j[q_j]$ , then

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p_j(x)y = (q_j - p_j)(x)u_j[q_j](x, t) & (x, t) \in (0, \ell) \times (0, T), \\ y(Q_j, t) = 0, & t \in (0, T), \\ y(x, 0) = 0, \partial_t y(x, 0) = 0, & x \in (0, \ell), \end{cases} \quad (11)$$

and noting that  $T > 0$  is sufficiently large, we can apply Lemma 2 since  $\partial_x y(Q_j, t) = 0$  for all  $t \in (0, T)$ ,  $u_j[q_j] \in H^1(0, T; L^\infty(\Lambda))$  and  $|u_j^0(x)| \geq r > 0$  on  $e_j$ . We obtain that  $p_j \equiv q_j$  on  $e_j$  and consequently  $u_j[p_j](x, t) = u_j[q_j](x, t)$  in  $e_j \times (0, T_1)$ , where  $T_1 \in (0, T)$  is some constant.

Therefore, for any segment  $e$  with the end-points  $P$  and  $Q$  such that  $Q \in \Pi_2 \setminus \{Q_{N_2}\}$ , we see that  $p = q$  on  $e$  and  $(u[p]|_e)(P, t) = (u[q]|_e)(P, t)$ ,  $(\partial_x u[p]|_e)(P, t) = (\partial_x u[q]|_e)(P, t)$  for  $0 < t < T_1$ . Let  $\Pi_1^2$  be all the interior node points  $P$  of segments of  $\Lambda$  having their other end-point in  $\Pi_2 \setminus \{Q_{N_2}\}$ . We note that  $\Pi_1^2 \subset \Pi_1$ . Applying the above argument to all the exterior end-points except for  $Q_{N_2}$ , we have

$$u[p]_j(P, t) = u[q]_j(P, t), \quad \partial_x u[p]_j(P, t) = \partial_x u[q]_j(P, t)$$

for each  $P \in \Pi_1^2$ ,  $0 < t < T_1$  and  $j \in \{1, \dots, N_3\}$ . Here by  $e_1, \dots, e_{N_3}$ , we enumerate the segments connecting a point in  $\Pi_1^2$  and a point in  $\Pi_2 \setminus \{Q_{N_2}\}$ .

**Step 2.** Let  $P \in \Pi_1$  be a given node such that  $n_P$  segments, say,  $e_1, \dots, e_{n_P}$  meet at  $P$  and  $e_1, \dots, e_{n_P-1}$  connect  $P$  with exterior end-points, say,  $Q_1, \dots, Q_{n_P-1} \in \Pi_2$  and

$$\begin{aligned} u[p]_j(P, t) &= u[q]_j(P, t), \\ \partial_x u[p]_j(P, t) &= \partial_x u[q]_j(P, t), \quad j \in \{1, \dots, n_P - 1\}, \quad 0 < t < T. \end{aligned} \quad (12)$$

Using the continuity (2) and the Kirchhoff law (3) at node  $P$ , we can deduce that

$$\begin{aligned} u[p]_{n_P}(P, t) &= u[q]_{n_P}(P, t), \\ \partial_x u[p]_{n_P}(P, t) &= \partial_x u[q]_{n_P}(P, t), \quad 0 < t < T. \end{aligned}$$

**Step 3.** Let  $\Lambda^2$  be the graph generated from  $\Lambda$  by removing  $e_1, \dots, e_{N_3}$ . Therefore, since  $T_1 > 0$  is still sufficiently large, we can apply the same argument as in Step 1 to the graph  $\Lambda^2$ .

We repeat this operation to obtain the sets  $\Lambda^3$ , then  $\Lambda^4, \dots, \Lambda^n$ . Hence, let  $L^k$  be the set of all the open segments of  $\Lambda_k$ ,  $\Pi_1^k$  the set of the interior node points of  $\Lambda_k$ ,  $\Pi_2^k$  the set of external endpoints of  $\Lambda_k$ . Setting  $\Lambda^1 = \Lambda$ , we note that  $L^1 = \{e_0, \dots, e_N\}$ ,  $\Pi_1^1 = \{P_1, \dots, P_{N_1}\}$ ,  $\Pi_2^1 = \{Q_1, \dots, Q_{N_2}\}$ .

By (2) and (3), we see that

$$\Pi_1^{k-1} \supset \Pi_1^k, \quad \forall k \in \mathbb{N}$$

and

$$\Lambda_k = L^k \cup \Pi_1^k \cup \Pi_2^k, \quad L^k \cap \Pi_1^k = L^k \cap \Pi_2^k = \Pi_1^k \cap \Pi_2^k = \emptyset, \quad \forall k \in \mathbb{N}.$$

In order to complete the proof, it is sufficient to prove there exists  $n \in \mathbb{N}$  such that

$$\Lambda_n = \emptyset. \quad (13)$$

Assume contrarily that  $\Lambda_n \neq \emptyset$  for all  $n \in \mathbb{N}$ . Since every segment with exterior end-point in  $\Pi_2 \setminus \{Q_{N_2}\}$ , can be removed (meaning that  $u[p] = u[q]$  on the segment) by the above operation, we obtain that there exists  $n_0 \in \mathbb{N}$  such that  $\Lambda_{n_0} = L^{n_0} \cup \Pi_1^{n_0}$ , i.e.,  $\Pi_2^{n_0} = \emptyset$ . Then  $\Lambda_{n_0}$  must be a closed cycle since it possesses no external endpoint. By assumption, there exist no closed cycles in a tree-shape network. This is a contradiction and thus the proof of (13), and therefore, the one of Theorem 1 is completed.  $\square$

### 3 Uniqueness for the inverse problem - Schrödinger and heat network cases

#### 3.1 Proof of Theorem 2 - Heat case.

We apply an argument similar to the proof of Theorem 4.7 in [12] which is based on the reduction of the inverse heat problem to an inverse wave problem by a kind of Laplace transform called the Reznitzkaya transform (e.g., [11], [15], [21]).

First we define an operator  $\Delta_\Lambda$  in  $L^2(\Lambda)$  by  $\Delta_\Lambda u = \partial_x^2 u_j$  in  $e_j$ , for all  $j \in \{0, 1, \dots, N\}$  with

$$\mathcal{D}(\Delta_\Lambda) = \{u = (u_0, \dots, u_N); u_j \in H^2(e_j), \partial_x u(Q) = 0 \text{ for } Q \in \Pi_2, u_j \text{ satisfying (2) and (3)}\}.$$

Here,  $e_j$  is oriented from  $I(e_j)$  to  $T(e_j)$  when defining  $\partial_x^2$ . Then, similarly to [8], we can prove that  $\Delta_\Lambda$  is self-adjoint and  $(\Delta_\Lambda u, u)_{L^2(\Lambda)} := \sum_{j=0}^N (\partial_x^2 u_j, u_j)_{L^2(e_j)} \geq 0$ . Therefore  $\Delta_\Lambda$  generates an analytic semigroup  $e^{t\Delta_\Lambda}$ ,  $t > 0$  (e.g., Pazy [19], Tanabe [23]). Since  $p \in L^\infty(\Lambda)$ , the perturbed operator  $\Delta_\Lambda + p$  generates an analytic semigroup (e.g., Theorem 2.1 in [19], p.80). Therefore by the semigroup theory (e.g. [19], [23]), we know that the solutions  $u[p](x, t)$  and  $u[q](x, t)$  of equation (4) are analytic in  $t$  for any fixed  $x \in \Lambda$ . More precisely,  $u[p], u[q] : (0, \infty) \rightarrow H^2(\Lambda)$  are analytic in  $t > 0$ .

By  $u^H[p]$  we denote the solution of the heat system (4) and by  $u^H[q]$  the corresponding solution when the potential is  $q$ . By the analyticity in  $t$  and the assumption in the theorem, we have

$$u^H[p](Q, t) = u^H[q](Q, t), \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall t > 0. \quad (14)$$

On the other hand, denote by  $\tilde{u}[p]$  the solution of the wave system

$$\begin{cases} \partial_t^2 u_j - \partial_x^2 u_j + p_j(x)u_j = 0, & \forall j \in \{0, 1, \dots, N\}, \forall (x, t) \in e_j \times (0, \infty), \\ \partial_x u[p](Q, t) = 0, & \forall Q \in \Pi_2, \forall t \in (0, \infty), \\ u[p](x, 0) = 0, \partial_t u(x, 0) = u^0(x), & \forall x \in \Lambda \end{cases} \quad (15)$$

and by  $\tilde{u}[q]$  the corresponding solution when the potential is  $q$ . Then we obtain (e.g., [15, pp.251-252]) that

$$\frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} \tilde{u}[p](x, \tau) d\tau$$

satisfies (4). The uniqueness of solution to equation (4) implies

$$u^H[p](x, t) = \frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} \tilde{u}[p](x, \tau) d\tau, \quad \forall x \in \Lambda, \forall t > 0$$

and the same equality with  $q$ . By assumption (14), we obtain

$$\frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} (\tilde{u}[p] - \tilde{u}[q])(Q, \tau) d\tau = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall t > 0.$$

By the change of variables  $s = \frac{1}{4t}$  and  $\tau^2 = \eta$ , we obtain

$$\int_0^\infty e^{-s\eta} (\tilde{u}[p] - \tilde{u}[q])(Q, \sqrt{\eta}) d\eta = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall s > 0$$

and the injectivity of the Laplace transform yields

$$(\tilde{u}[p] - \tilde{u}[q])(Q, \sqrt{\eta}) = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall \eta > 0. \quad (16)$$

Applying the same argument as in Section 2 for the wave system, we prove  $p = q$  in  $\Lambda$ . Thus the proof of Theorem 2 is completed.

### 3.2 Proof of Theorem 3 - Schrödinger case.

It is sufficient to prove the following lemma.

**Lemma 3** *Let  $r > 0$  and  $p \in L^\infty(0, \ell)$ ,  $f \in L^2(0, \ell)$  be real-valued, and  $T > 0$  be arbitrarily fixed. We consider a 1-d Schrödinger equation:*

$$\begin{cases} i\partial_t y - \partial_x^2 y + p(x)y = f(x)R(x, t), & \forall (x, t) \in (0, \ell) \times (0, T), \\ y(\ell, t) = 0, & \forall t \in (0, T), \\ y(x, 0) = 0, & \forall x \in (0, \ell), \end{cases}$$

where  $R \in H^1(0, T; L^\infty(0, \ell))$  satisfies  $|R(x, 0)| \geq r > 0$  a.e. in  $(0, \ell)$ .

If  $\partial_x y(\ell, t) = 0$  for all  $t \in (0, T)$ , then we have  $f = 0$  in  $(0, \ell)$  and  $y = 0$  in  $(0, \ell) \times (0, T)$ .

Using the same method as the one for the proof of Lemma 2, this lemma is proved by means of the following Carleman estimate:

**Lemma 4** *For  $x_0 \notin [0, \ell]$  and  $\beta > 0$  arbitrarily fixed, we set*

$$Sv = i\partial_t v - \partial_x^2 v, \quad \varphi(x, t) = e^{\gamma(|x-x_0|^2 - \beta t^2)}, \quad (x, t) \in (0, \ell) \times (0, T).$$

Then there exists a constant  $\gamma_0 > 0$  such that for arbitrary  $\gamma \geq \gamma_0$  we can choose  $s_0 > 0$  satisfying, for a constant  $C > 0$ ,

$$\int_0^T \int_0^\ell (s|\partial_x v|^2 + s^3|v|^2) e^{2s\varphi} dx dt \leq C \int_0^T \int_0^\ell |Sv|^2 e^{2s\varphi} dx dt$$

for all  $s > s_0$  and all  $v \in L^2(0, T; H_0^2(0, \ell)) \cap H_0^1(0, T; L^2(0, \ell))$ .

This is a Carleman estimate with regular weight function  $\gamma(|x-x_0|^2 - \beta t^2)$  and for the proof, we refer to e.g. [26, Lemma 2.1] (see also [27]). Concerning a Carleman estimate for Schrödinger equation in a bounded domain  $\Omega \subset \mathbb{R}^n$  with singular weight function  $\varphi$ , we can refer for example to [3, 18].

On the basis of this lemma, the proof of Lemma 3 is done by a usual method by Bukhgeim and Klibanov [4] by using the extension of  $y$  to  $-T < t < 0$  by  $y(\cdot, t) = \overline{y(\cdot, -t)}$  and a cut-off argument. We omit the details of the proof.

## 4 Observability in the wave network

The proof of the stability result will rely strongly on the classical result of observability that we are now presenting and proving. One should specifically mention the survey [28] and the books [7], [14], where the question of observability in networks of strings (or wave equations) is widely explored in different cases.

We concentrate here on the case where the observation available comes from all but one external nodes, in a setting with a system of wave equations with potential. Since most of the literature on string networks focus only on the wave equation without lower order terms (see [14] or [7] for instance), we detail here how to obtain the observability result for the wave equation with potential. In some other cases, we can prove the observability inequality directly by a global Carleman estimate (e.g. [1]).

**Theorem 5 (Observability inequality)** *On the tree-shaped network  $\Lambda$ , assuming  $p \in L^\infty(\Lambda)$ , let us consider the system of 1-d wave equations under the continuity and Kirchhoff law's assumptions (2) and (3):*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + p(x)u = 0, & \text{in } \Lambda \times (0, T), \\ u(Q, t) = 0, & \text{in } (0, T), \forall Q \in \Pi_2, \\ u_j(P, t) = u_k(P, t), & \text{in } (0, T), \forall P \in \Pi_1, \forall j, k \in \{1, \dots, n_P\}, \\ [\partial_x u(t)]_P = 0, & \text{in } (0, T), \forall P \in \Pi_1, \\ u(x, 0) = 0, \quad \partial_t u(x, 0) = a(x), & \text{in } \Lambda, \end{cases} \quad (17)$$



Then there exists a minimal time  $T_0$  such that for all  $T > T_0$ , the observability estimate

$$\int_{\Lambda} |a(x)|^2 dx \leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j, t)|^2 dt \quad (18)$$

holds for a solution  $u$  of (17).

**Proof of Theorem 5.** Let  $v$  be the solution of the system

$$\begin{cases} \partial_t^2 v - \partial_x^2 v = -pu & \forall (x, t) \in \Lambda \times (0, T), \\ v(Q, t) = 0, & \forall Q \in \Pi_2, t \in (0, T), \\ v_j(x, 0) = 0, \partial_t v_j(x, 0) = 0, & \forall j \in \{0, 1, \dots, N\}, \quad x \in e_j, \end{cases}$$

under conditions (2) and (3). Then (9) in Lemma 1 and  $p \in L^\infty(\Lambda)$  yields

$$\sum_{j=1}^{N_2} \int_0^T |\partial_x v_j(Q_j, t)|^2 dt \leq C \int_0^T \int_{\Lambda} |pu|^2 dx dt \leq C \int_0^T \int_{\Lambda} |u|^2 dx dt. \quad (19)$$

Setting  $w = u - v$ , we still have (2) and (3) satisfied by  $w$ , along with the following equation

$$\begin{cases} \partial_t^2 w - \partial_x^2 w = 0 & \forall (x, t) \in \Lambda \times (0, T), \\ w(Q, t) = 0, & \forall Q \in \Pi_2, t \in (0, T), \\ w_j(x, 0) = 0, \partial_t w_j(x, 0) = a(x), & \forall x \in \Lambda. \end{cases}$$

Therefore, using a classical observability inequality in the case where  $p = 0$  (e.g., [7, 14]), we have

$$\int_{\Lambda} |a(x)|^2 dx \leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x w_j(Q_j, t)|^2 dt.$$

Hence, by (19), we have

$$\begin{aligned} \int_{\Lambda} |a(x)|^2 dx &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j, t)|^2 dt + C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x v_j(Q_j, t)|^2 dt \\ &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j, t)|^2 dt + C \int_0^T \int_{\Lambda} |u|^2 dx dt. \end{aligned} \quad (20)$$

Therefore a usual compactness-uniqueness argument yields the observability inequality (18). Indeed, if (18) is not satisfied, then we can assume that there exists  $a^n \in L^2(\Lambda)$ ,  $n \in \mathbb{N}$  such that

$$\|a^n\|_{L^2(\Lambda)} = 1, \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^n(Q_j, t)|^2 dt = 0. \quad (21)$$

Using the energy estimate (8) of Lemma 1 on the solution  $u^n$  of system (17) with initial data  $a^n$ , we obtain

$$\|u^n(t)\|_{H_0^1(\Lambda)}^2 = \|\partial_x u^n(t)\|_{L^2(\Lambda)}^2 \leq C \|a^n\|_{L^2(\Lambda)}^2 \leq C.$$

Since the embedding  $H_0^1(\Lambda) \subset L^2(\Lambda)$  is compact, we can extract a subsequence, denoted again by the same notation and we have  $(u^n)_{n \in \mathbb{N}^*}$  convergent in  $L^2(\Lambda)$ . Therefore, using (20), we obtain

$$\begin{aligned} \int_{\Lambda} |a^n - a^m|^2 dx &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^n(Q_j, t)|^2 dt + C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^m(Q_j, t)|^2 dt \\ &\quad + C \int_0^T \int_{\Lambda} |u^n - u^m|^2 dx dt \end{aligned}$$

so that (21) and  $\lim_{n, m \rightarrow \infty} \|u^n - u^m\|_{L^2(\Lambda)} = 0$  imply  $\lim_{n, m \rightarrow \infty} \|a^n - a^m\|_{L^2(\Lambda)}^2 = 0$ . Consequently, there exists a limit  $a_0$  such that  $\lim_{n \rightarrow +\infty} a^n = a_0$  in  $L^2(\Lambda)$  and from (21), we have  $\|a_0\|_{L^2(\Lambda)} = 1$ .

Moreover, the solution  $u[a_0]$  of system (17) with initial data  $a_0$  is such that

$$\partial_x u_j^m[a_0](Q, t) = 0, \quad \forall t \in (0, T), \forall Q \in \Pi_2.$$

Hence we apply a classical unique continuation result for a wave equation to obtain that  $u[a_0]$  vanishes everywhere so that  $a_0 = 0$ , which contradicts  $\|a_0\|_{L^2(\Lambda)} = 1$ . Here, the unique continuation can be proved for instance by a Carleman estimate (e.g. [11], [13]). This ends the proof of Theorem 5.

## 5 Proof of the stability for the wave network inverse problem

This section is devoted to the proof of Theorem 4. The proof relies on a compactness-uniqueness argument and the observability estimate (Theorem 5) on the whole network.

Let us denote by  $u[p]$  the solution of (1) under the assumptions (2) and (3). Henceforth we always assume the conditions (2) and (3). We consider  $y = \partial_t(u[p] - u[q])$  that satisfy

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + q(x)y = (q-p)\partial_t u[p] & \forall (x,t) \in \Lambda \times (-T,T), \\ y(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ y(x,0) = 0, \partial_t y(x,0) = (q-p)u^0(x), & \forall x \in \Lambda, \end{cases} \quad (22)$$

We define  $\psi$  and  $\phi$  as the solutions of

$$\begin{cases} \partial_t^2 \psi - \partial_x^2 \psi + q(x)\psi = (q-p)\partial_t u[p] & \forall (x,t) \in \Lambda \times (-T,T), \\ \psi(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ \psi(x,0) = 0, \partial_t \psi(x,0) = 0, & \forall x \in \Lambda, \end{cases} \quad (23)$$

and

$$\begin{cases} \partial_t^2 \phi - \partial_x^2 \phi + q(x)\phi = 0 & \forall (x,t) \in \Lambda \times (-T,T), \\ \phi(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ \phi(x,0) = 0, \partial_t \phi(x,0) = (q-p)u^0(x), & \forall x \in \Lambda. \end{cases} \quad (24)$$

such that  $y = \psi + \phi$ . We can apply Theorem 5 to equation (24) so that

$$\int_{\Lambda} |(q-p)u^0|^2 dx \leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x \phi_j(Q_j, t)|^2 dt. \quad (25)$$

On the other hand, a regularity result of Lemma 1 applied to a time derivative of equation (23) gives

$$\begin{aligned} \sum_{j=1}^{N_2} \|\partial_x \psi_j(Q_j)\|_{H^1(0,T)}^2 &\leq C (\|(q-p)\partial_t^2 u[p]\|_{L^1(0,T,L^2(\Lambda))}^2 + \|(q-p)u^1\|_{L^2(\Lambda)}^2) \\ &\leq 2CK^2 \|q-p\|_{L^2(\Lambda)}^2 \end{aligned} \quad (26)$$

as soon as we have  $u[p] \in H^2(0,T,L^\infty(\Lambda))$  which yields  $\partial_t u[p] \in C([0,T];L^\infty(\Lambda))$  so that  $u^1 \in L^\infty(\Lambda)$  with  $\|u[p]\|_{H^2(0,T,L^\infty(\Lambda))} \leq K$ . The compact embedding  $H^1(0,T) \subset L^2(0,T)$  allows then to write that the operator  $\Psi : L^2(\Lambda) \rightarrow L^2(0,T)$  defined by

$$\Psi(p-q)(t) = \sum_{j=1}^{N_2} \partial_x \psi_j(Q_j, t), \quad 0 < t < T$$

is compact.

Therefore, since we have  $|u^0(x)| \geq r > 0$  almost everywhere in  $\Lambda$ , by (25) and (26), we obtain

$$\begin{aligned} \|q-p\|_{L^2(\Lambda)} &\leq C \int_{\Lambda} |(q-p)u^0|^2 dx \leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x \phi_j(Q_j, t)|^2 dt \\ &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x y_j(Q_j, t)|^2 dt + C \sum_{j=1}^{N_2} \int_0^T |\partial_x \psi_j(Q_j, t)|^2 dt \\ &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x y_j(Q_j, t)|^2 dt + C \|\Psi(q-p)\|_{L^2(0,T)}^2 \\ &\leq C \sum_{j=1}^{N_2-1} \|\partial_x u_j[p](Q_j) - \partial_x u_j[q](Q_j)\|_{H^1(0,T)}^2 + C \|\Psi(q-p)\|_{L^2(0,T)}^2. \end{aligned} \quad (27)$$

We aim at proving that we can get rid of the second term on the right-hand side of the last estimate in order to obtain (6). Again, a compactness-uniqueness argument will be the key and it relies here on the compactness of  $\Psi$  and the uniqueness result of Theorem 1.

Indeed, we set  $f = q - p$ . We assume that

$$\|f\|_{L^2(\Lambda)} \leq C \sum_{j=1}^{N_2-1} \|\partial_x y_j(Q_j)\|_{L^2(0,T)},$$

which is equivalent to (6), does not hold. Then one can assume that there exists  $f^n \in L^2(\Lambda)$ ,  $n \in \mathbb{N}$  such that

$$\|f^n\|_{L^2(\Lambda)} = 1, \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sum_{j=1}^{N_2-1} \|\partial_x y_j^n(Q_j)\|_{L^2(0,T)} = 0. \quad (28)$$

First, since the sequence  $(f^n)_{n \in \mathbb{N}}$  is bounded in  $L^2(\Lambda)$ , we can extract a subsequence denoted again by  $(f^n)_{n \in \mathbb{N}}$  such that it converges towards some  $f^0 \in L^2(\Lambda)$  weakly in  $L^2(\Lambda)$ . Since  $\Psi$  is a compact operator, we obtain therefore the strong convergence result

$$\lim_{n, m \rightarrow \infty} \|\Psi(f^n) - \Psi(f^m)\|_{L^2(0,T)} = 0. \quad (29)$$

Then, from (27) we can write

$$\|f^n - f^m\|_{L^2(\Lambda)} \leq C \sum_{j=1}^{N_2-1} \|\partial_x y_j^n(Q_j)\|_{L^2(0,T)} + C \sum_{j=1}^{N_2-1} \|\partial_x y_j^m(Q_j)\|_{L^2(0,T)} + C \|\Psi(f^n) - \Psi(f^m)\|_{L^2(\Lambda)}^2$$

and deduce from (28) and (29) that  $\lim_{n, m \rightarrow \infty} \|f^n - f^m\|_{L^2(\Lambda)} = 0$ , so that  $\lim_{n \rightarrow \infty} \|f^n - f^0\|_{L^2(\Lambda)} = 0$  with

$$\|f^0\|_{L^2(\Lambda)} = 1. \quad (30)$$

Moreover, using the trace estimate (9) of Lemma 1 for the solution  $y^n$  of system (22) with initial data  $f^n u^0$  and source term  $f^n \partial_t u[p]$ , we obtain

$$\sum_{j=1}^{N_2-1} \|\partial_x y_j^n(Q_j)\|_{L^2(0,T)}^2 \leq C (\|f^n u^0\|_{L^2(\Lambda)}^2 + \|f^n \partial_t u[p]\|_{L^1(0,T;L^2(\Lambda))}^2) \leq 2CK^2 \|f^n\|_{L^2(\Lambda)}.$$

Thus we can write

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{N_2-1} \|\partial_x y_j^n(Q_j) - \partial_x y_j^0(Q_j)\|_{L^2(0,T)}^2 \leq 2CK^2 \lim_{n \rightarrow \infty} \|f^n - f^0\|_{L^2(\Lambda)} = 0,$$

which, combined with (28), gives

$$\partial_x y_j^0(Q, t) = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall t \in (0, T).$$

We finally apply Theorem 1 and obtain  $f^0 = 0$  in  $L^2(\Lambda)$ , which contradicts (30). Thus the proof of Theorem 4 is complete.

## References

- [1] L. Baudouin, E. Crépeau, and J. Valein. Global Carleman estimate on a network for the wave equation and application to an inverse problem. *Math. Control Relat. Fields*, 1(3):307–330, 2011.
- [2] L. Baudouin, A. Mercado, and A. Osses. A global Carleman estimate in a transmission wave equation and application to a one-measurement inverse problem. *Inverse Problems*, 23(1):257–278, 2007.
- [3] L. Baudouin and J.-P. Puel. Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Problems*, 18(6):1537–1554, 2002.
- [4] A. L. Bukhgeim and M. V. Klivanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.
- [5] R. Dáger. Observation and control of vibrations in tree-shaped networks of strings. *SIAM J. Control Optim.*, 43(2):590–623 (electronic), 2004.

- [6] R. Dáger and E. Zuazua. Controllability of star-shaped networks of strings. In *Mathematical and numerical aspects of wave propagation (Santiago de Compostela, 2000)*, pages 1006–1010. SIAM, Philadelphia, PA, 2000.
- [7] R. Dáger and E. Zuazua. *Wave propagation, observation and control in 1-d flexible multi-structures*, volume 50 of *Mathématiques & Applications (Berlin)*. Springer-Verlag, Berlin, 2006.
- [8] L. I. Ignat, A. F. Pazoto, and L. Rosier. Inverse problem for the heat equation and the Schrödinger equation on a tree. *Inverse Problems*, 28(1):015011, 30, 2012.
- [9] O. Y. Imanuvilov and M. Yamamoto. Global Lipschitz stability in an inverse hyperbolic problem by interior observations. *Inverse Problems*, 17(4):717–728, 2001. Special issue to celebrate Pierre Sabatier’s 65th birthday (Montpellier, 2000).
- [10] O. Y. Imanuvilov and M. Yamamoto. Global uniqueness and stability in determining coefficients of wave equations. *Comm. Partial Differential Equations*, 26(7-8):1409–1425, 2001.
- [11] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2006.
- [12] M. V. Klibanov. Inverse problems and Carleman estimates. *Inverse Problems*, 8(4):575–596, 1992.
- [13] M. V. Klibanov and A. Timonov. *Carleman estimates for coefficient inverse problems and numerical applications*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 2004.
- [14] J. Lagnese, G. Leugering, and E. J. P. G. Schmidt. *Modeling, analysis and control of dynamic elastic multi-link structures*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.
- [15] M.M. Lavrentev, V.G. Romanov, and S.P. Shishatskiĭ. *Ill-posed Problems of Mathematical Physics and Analysis*. Translations of Mathematical Monographs. American Mathematical Society, Providence, R.I., 1986.
- [16] J.-L. Lions. *Contrôlabilité exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1. Contrôlabilité exacte*, volume RMA 8. Masson, 1988.
- [17] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.
- [18] A. Mercado, A. Osses, and L. Rosier. Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights. *Inverse Problems*, 24(1):015017, 18, 2008.
- [19] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [20] J.-P. Puel and M. Yamamoto. Generic well-posedness in a multidimensional hyperbolic inverse problem. *J. Inverse Ill-Posed Probl.*, 5(1):55–83, 1997.
- [21] V. G. Romanov. *Inverse problems of mathematical physics*. VNU Science Press, b.v., Utrecht, 1987. With a foreword by V. G. Yakhno, Translated from the Russian by L. Ya. Yuzina.
- [22] E. J. P. G. Schmidt. On the modelling and exact controllability of networks of vibrating strings. *SIAM J. Control Optim.*, 30(1):229–245, 1992.
- [23] Hiroki Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979. Translated from the Japanese by N. Mugibayashi and H. Haneda.
- [24] M. Yamamoto. Uniqueness and stability in multidimensional hyperbolic inverse problems. *J. Math. Pures Appl. (9)*, 78(1):65–98, 1999.
- [25] M. Yamamoto and X. Zhang. Global uniqueness and stability for a class of multidimensional inverse hyperbolic problems with two unknowns. *Appl. Math. Optim.*, 48(3):211–228, 2003.
- [26] G. Yuan and M. Yamamoto. Lipschitz stability in inverse problems for a Kirchhoff plate equation. *Asymptot. Anal.*, 53(1-2):29–60, 2007.
- [27] G. Yuan and M. Yamamoto. Carleman estimates for the Schrödinger equation and applications to an inverse problem and an observability inequality. *Chin. Ann. Math. Ser. B*, 31(4):555–578, 2010.
- [28] E. Zuazua. *Control and stabilization of waves on 1-d networks*, volume Modelling and Optimisation of Flows on Networks of *Lecture Notes in Mathematics, CIME foundation subseries*. Springer, 2011.